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Donald L. Kunz, Aeromechanics Laboratory
AVRADCOM Research and Technical Laboratories
Ames Researc h Center, Moffett Field, California

National Aeronautics and
Space Administration

Ames Research Center
Moffett Field, California 94035

United States Army
Aviation Research and
Development Command
St. Louis, Missouri 63166

PRODUCTS OF MULTIPLE FOURIER SERIES WITH APPLICATION TO THE MULTIBLADE TRANSFORMATION

Donald L. Kunz

Research Scientist, Aeromechanics Laboratory

U.S. Army R&T Laboratories (AVRADCOM)

SUMMARY

A relatively simple and systematic method for forming the products of multiple Fourier series using tensor-like operations is demonstrated. This symbolic multiplication can be performed for any arbitrary number of series, and only the coefficients of one series need to be known. The application of this methodology to the transformation of a set of linear differential equations with periodic coefficients from a rotating coordinate system to a nonrotating system is also demonstrated. It is shown that using Fourier operations to perform this transformation make it easily understood, simple to apply, and generally applicable.

INTRODUCTION

The harmonic balance method is a useful tool for obtaining the forced response of systems governed by differential equations with periodic coefficients. However, for a set of linear equations, the manual application of this method tends to become quite unwieldy as the number of harmonics, or the number of degrees of freedom, increases. In reference 1, Peters and Ormiston develop matrix methods that will perform the Fourier operations necessary to solve linear equations, eliminating the need for hand calculations involving Fourier series. Several investigations (refs. 1-3) have used this methodology to successfully solve linear response problems.

As opposed to linear problems in which products of two Fourier series are encountered, nonlinear problems retaining only second-order nonlinearities contain products of three series. Reference 4, which considers such a problem, handles products of three series by multiplying the first two series in one step and the third in a separate step. This presupposes that the coefficients of both of the first two series are known, as they are in that particular application. However, if the coefficients of only one series are known, this method of forming the product of multiple Fourier series cannot be used.

The alternative to performing successive muitiplications (of two series to form the product of multiple series) is to derive an array analogous to the Fourier product matrix of reference 1. It is the objective of this Memorandum to demonstrate the procedure for forming such an array for any arbitrary number of series. In addition, the application of Fourier series to performing the multiblade transformation (refs. 5 and 6) is described. This application uses summation relations, Fourier derivatives, and products of two and three series. Some of these operations can be found in reference 1, but a complete set has been included in this report.

LINEAR OPERATIONS

In reference 1, Peters and Ormiston present the identities and linear operations that are required to develop Fourier products, and to apply the generalized harmonic balance method to the solution of differential equations. For reasons of completeness and to establish a consistent notation within this report, these identities and linear operations are repeated below.

Basic identities- Consider a function $f(\psi)$ with period 2π. The Fourier coefficients of $f(\psi)$ are defined as follows:

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(\psi)\cos(n\psi)d\psi \quad (1)$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(\psi)\sin(n\psi)d\psi \quad (2)$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\psi)d\psi \quad (3)$$

$$b_0 = 0 \quad (4)$$

In periodic systems having b equally spaced, identical, time-lagging components (e.g., helicopter rotors, turbine compressor rotors, and wind turbines), the following summation relations are frequently useful.

$$\frac{1}{b}\sum_{k=1}^{b} \cos(n\psi_k) = \begin{cases} \cos(n\psi_0), & \text{if } n \text{ is an integer multiple of } b \\ 0, & \text{if } n \text{ is not an integer multiple of } b \end{cases} \quad (5)$$

$$\frac{1}{b}\sum_{k=1}^{b} \sin(n\psi_k) = \begin{cases} \sin(n\psi_0), & \text{if } n \text{ is an integer multiple of } b \\ 0, & \text{if } n \text{ is not an integer multiple of } b \end{cases} \quad (6)$$

$$\frac{1}{b}\sum_{k=1}^{b} (-1)^k \cos(n\psi_k) = \begin{cases} \cos(n\psi_0), & \text{if } n \text{ is an odd integer multiple of } \frac{b}{2} \\ 0, & \text{if } n \text{ is not an odd integer multiple of } \frac{b}{2} \end{cases} \quad (7)$$

$$\frac{1}{b}\sum_{k=1}^{b} (-1)^k \sin(n\psi_k) = \begin{cases} \sin(n\psi_0), & \text{if } n \text{ is an odd integer multiple of } \frac{b}{2} \\ 0, & \text{if } n \text{ is not an odd integer multiple of } \frac{b}{2} \end{cases} \quad (8)$$

where $\psi_k = \psi_0 + 2\pi(k - 1)/b$. The identities in equations (7) and (8) do not appear in reference 1, but can be found in reference 6. They arise only for periodic systems in which b is even. Equations (5) and (6), on the other hand, are applicable to systems in which b is either even or odd.

Phase change- To express the phase-shifted function $f(\psi + \delta)$ as a function of $f(\psi)$, let $g(\psi) = f(\psi + \delta)$ and express f and g as Fourier series, where N is a nonnegative integer.

$$f(\psi) = \sum_{n=0}^{N} [a_n \cos(n\psi) + b_n \sin(n\psi)] \tag{9}$$

$$g(\psi) = \sum_{n=0}^{N} [A_n \cos(n\psi) + B_n \sin(n\psi)] \tag{10}$$

Writing g in terms of f and expanding,

$$g(\psi) = \sum_{n=0}^{N} \{a_n \cos[n(\psi + \delta)] + b_n \sin[n(\psi + \delta)]\} \tag{11}$$

$$\begin{aligned} g(\psi) = \sum_{n=0}^{N} \{&[a_n \cos(n\delta) + b_n \sin(n\delta)]\cos(n\psi) \\ &+ [-a_n \sin(n\delta) + b_n \cos(n\delta)]\sin(n\psi)\} \end{aligned} \tag{12}$$

The A_n and B_n are then

$$\left.\begin{aligned} A_n &= a_n \cos(n\delta) + b_n \sin(n\delta) \\ B_n &= -a_n \sin(n\delta) + b_n \cos(n\delta) \end{aligned}\right\} \tag{13}$$

This relationship can be written in matrix form as

$$\begin{Bmatrix} A_n \\ B_n \end{Bmatrix}_g = [\Phi(\delta)] \begin{Bmatrix} a_n \\ b_n \end{Bmatrix}_f \tag{14}$$

where

$$[\Phi(\delta)] = \left[\begin{array}{cccc|cccc} 1 & & & & 0 & & & \\ & \cos\delta & & \underline{0} & & \sin\delta & & \underline{0} \\ & & \ddots & & & & \ddots & \\ & \underline{0} & & & & \underline{0} & & \\ & & & \cos(N\delta) & & & & \sin(n\delta) \\ \hline 0 & & & & 1 & & & \\ & -\sin\delta & & \underline{0} & & -\cos\delta & & \underline{0} \\ & & \ddots & & & & \ddots & \\ & \underline{0} & & & & \underline{0} & & \\ & & & -\sin(N\delta) & & & & \cos(N\delta) \end{array}\right] \tag{15}$$

In the next section, a matrix operation which forms the derivative of a Fourier series will be discussed. That derivative matrix is just a special case of equation (15), where the phase angle δ equals 90°.

Derivative- In order to apply the harmonic balance method to a system of differential equations, it is necessary to be able to take the derivative of a Fourier series. Given a seri $f(\psi)$ and its derivative,

$$f(\psi) = \sum_{n=0}^{N} [a_n \cos(n\psi) + b_n \sin(n\psi)] \tag{16}$$

$$\frac{\partial f}{\partial \psi} = \sum_{n=0}^{N} [-na_n \sin(n\psi) + nb_n \sin(n\psi)] \tag{17}$$

set $g(\psi)$ equal to the derivative of $f(\psi)$.

$$g(\psi) = \sum_{n=0}^{N} [A_n \cos(n\psi) + B_n \sin(n\psi)] \tag{18}$$

Then,

$$\begin{aligned} A_n &= nb_n \\ B_n &= -na_n \end{aligned} \tag{19}$$

Equations (19) can be written in matrix form as

$$\begin{Bmatrix} A_n \\ B_n \end{Bmatrix}_g = [D] \begin{Bmatrix} a_n \\ b_n \end{Bmatrix}_f \tag{20}$$

where

$$[D] = \left[\begin{array}{cccc:cccc} & & & & 0 & & & \\ & \underline{0} & & & & 1 & & \underline{0} \\ & & & & & & 2 & \\ & & & & & \underline{0} & & \ddots \\ & & & & & & & N \\ \hdashline 0 & & & & & & & \\ & -1 & & \underline{0} & & & & \\ & & -2 & & & & \underline{0} & \\ & \underline{0} & & \ddots & & & & \\ & & & -N & & & & \end{array}\right] \tag{21}$$

By extension of the concept used to develop equation (21), the derivative matrix used to calculate the kth derivative of $f(\psi)$ can be obtained.

$$\begin{Bmatrix} A_n \\ B_n \end{Bmatrix}_g = [D^k] \begin{Bmatrix} a_n \\ b_n \end{Bmatrix}_f \tag{22}$$

where

$$[D^k] = [D]^k$$

Peters and Ormiston, in reference 1, discuss the case where $f(\psi)$ is modulated by the harmonic function $e^{i\omega t}$. This concept can be generalized for the case where the Fourier coefficients of $f(\psi)$ are also functions of ψ.

$$f(\psi) = \sum_{n=0}^{N} [a_n(\psi)\cos(n\psi) + b_n(\psi)\sin(n\psi)] \tag{23}$$

Taking the derivative of f and calling it g,

$$g(\psi) = \frac{\partial f}{\partial \psi} = \sum_{n=0}^{N} \left[\frac{\partial a_n}{\partial \psi} + nb_n \cos(n\psi) + \frac{\partial b_n}{\partial \psi} - na_n \sin(n\psi)\right] \tag{24}$$

$$\begin{aligned} A_n &= \frac{\partial a_n}{\partial \psi} + nb_n \\ B_n &= \frac{\partial b_n}{\partial \psi} - na_n \end{aligned} \tag{25}$$

In matrix form, where $[I]$ is the identity matrix,

$$\begin{Bmatrix} A_n \\ B_n \end{Bmatrix}_g = \left[[I] \frac{\partial}{\partial \psi} + [D]\right] \begin{Bmatrix} a_n \\ b_n \end{Bmatrix}_f \tag{26}$$

For the kth derivative of $f(\psi)$

$$\begin{Bmatrix} A_n \\ B_n \end{Bmatrix}_g = \left[[I] \frac{\partial}{\partial \psi} + [D]\right]^k \begin{Bmatrix} a_n \\ b_n \end{Bmatrix}_f \tag{27}$$

One of the uses of equation (27) will be seen during the discussion of the method of multiblade coordinates.

Products of Fourier Series

In this section, a method for forming products of any number of Fourier series using matrix algebra is described. The Fourier product is one of the most essential operations required to apply the generalized harmonic balance method to the solution of differential equations. Reference 1 discusses the technique used to multiply two series together, but does not attempt to extend it to multiple products. In order to form a basis of understanding of the methodology and to establish consistent notation,

products of two Fourier series are reviewed. Then, it is shown how to extend the method to products of several series, using the product of three series as an example.

Products of two Fourier series- Consider two Fourier series expansions, $f(\psi)$ and $X(\psi)$, where the coefficients of $f(\psi)$ are known and I and J are nonnegative integers.

$$f(\psi) = \sum_{i=0}^{I} [a_i \cos(i\psi) + b_i \sin(i\psi)] \tag{28}$$

$$X(\psi) = \sum_{j=0}^{J} [x_{jc} \cos(j\psi) + x_{js} \sin(j\psi)] \tag{29}$$

The product of f and X is defined to be $Z(\psi)$, where

$$Z(\psi) = \sum_{n=0}^{N} [z_{nc} \cos(n\psi) + z_{ns} \sin(n\psi)] \tag{30}$$

where N is a nonnegative integer. Expanding Z in terms of f and X,

$$\begin{aligned} Z(\psi) = \frac{1}{2} \sum_{i=0}^{I} \sum_{j=0}^{J} \{ & a_i x_{jc}[\cos(i + j)\psi + \cos(i - j)\psi] + a_i x_{js}[\sin(i + j)\psi \\ & + \sin(i - j)\psi] + b_i x_{js}[-\cos(i + j)\psi + \cos(i - j)\psi] \\ & + b_i x_{jc}[\sin(i + j)\psi + \sin(i - j)\psi]\} \end{aligned} \tag{31}$$

The object now is to develop a way to express equations (30) and (31) in the form

$$\begin{Bmatrix} z_{nc} \\ z_{ns} \end{Bmatrix} = [P_2(f)] \begin{Bmatrix} x_{jc} \\ x_{js} \end{Bmatrix} \tag{32}$$

where $[P_2(f)]$ is the Fourier product matrix for the product of two series.

In order to form equation (32) the equivalent harmonics in equations (30) and (31) must be identified and separated such that $i + j = n$ and $i - j = n$. To this end, the following matrices are defined (α representing either a or b):

$$Q^{+-}_{nj}(\alpha) = \left.\begin{cases} \alpha_{n-j} & ; \quad 0 \le n - j \le I \\ 0 & ; \quad n - j < 0,\ n - j > I \end{cases}\right\} i + j = n \tag{33}$$

$$Q^{--}_{nj}(\alpha) = \left.\begin{cases} \alpha_{-n-j} & ; \quad 0 \le -n - j \le I \\ 0 & ; \quad -n - j < 0,\ -n - j > I \end{cases}\right\} -(i + j) = n \tag{34}$$

$$Q_{nj}^{++}(\alpha) = \left.\begin{cases} \alpha_{n+j} & ; \quad 0 \le n + j \le I \\ 0 & ; \quad n + j < 0,\ n + j > I \end{cases}\right\} i - j = n \tag{35}$$

$$Q_{nj}^{-+}(\alpha) = \left.\begin{cases} \alpha_{-n+j} & ; \quad 0 \le n + j \le I \\ 0 & ; \quad -n + j < 0,\ -n + j > I \end{cases}\right\} -(i - j) = n \tag{36}$$

Note that the nth row of the Q-matrices represents the nth harmonic of Z, and the jth column will be multiplied by the jth harmonic component of X. The harmonic component of α that goes in the nth row and jth column of the Q-matrices is calculated according to equations (33)-(36). If the subscript of α is called i, it can be seen the Q-matrices provide a means of identifying like harmonics. It is important to understand how this works for the product of two series since the same principle holds for multiple products, but is much more difficult to produce without some sort of pattern to follow.

The Q-matrices can now be combined to form the cosines and sines of like harmonics. The following substitutions are used:

$$\sum_{i=0}^{I} \alpha_i \cos(i + j)\psi = \sum_{m=0}^{N} \bar{I}_{nm}[Q_{mj}^{+-}(\alpha) + Q_{mj}^{--}(\alpha)] \; ; \qquad n = i + j \tag{37}$$

$$\sum_{i=0}^{I} \alpha_i \cos(i - j)\psi = \sum_{m=0}^{N} \bar{I}_{nm}[Q_{mj}^{++}(\alpha) + Q_{mj}^{-+}(\alpha)] \; ; \qquad n = i - j \tag{38}$$

$$\sum_{i=0}^{I} \alpha_i \sin(i + j)\psi = Q_{nj}^{+-}(\alpha) - Q_{nj}^{--}(\alpha) \qquad ; \qquad n = i + j \tag{39}$$

$$\sum_{i=0}^{I} \alpha_i \sin(i - j)\psi = Q_{nj}^{++}(\alpha) - Q_{nj}^{-+}(\alpha) \qquad ; \qquad n = i - j \tag{40}$$

where α may represent either a or b, and where $[\bar{I}]$ is a correction matrix

$$[\bar{I}] = \begin{bmatrix} \frac{1}{2} & & & & \\ & 1 & & \underline{0} & \\ & & 1 & & \\ & \underline{0} & & \ddots & \\ & & & & 1 \end{bmatrix} \tag{41}$$

In equations (37)-(40), the significance of the pattern of signs must be stressed. By changing from (i + j) to (i - j), the second sign in the Q-matrix superscript changes from minus to plus. Because of the way in which the Q-matrices were originally defined, the second superscript reflects how the jth harmonic of X contributes to the definition of the subscript of α. Thus, it is logical that when j changes sign, the second subscript should also change. In addition, the signs of the second matrices in equations (37) and (38), which define the cosines, are positive; while in equations (39) and (40), which define the sines, the same matrices

are negative. These sign patterns will be valuable when the multiple product matrices are formed.

All of the elements required to form the Fourier product matrix for products of two series have now been developed, and $[P_2(f)]$ itself can now be formed. First, partition $[P_2(f)]$ such that

$$\begin{Bmatrix} z_{nc} \\ \hline z_{ns} \end{Bmatrix} = \left[\begin{array}{c|c} A^c & B^s \\ \hline B^c & A^s \end{array}\right] \begin{Bmatrix} x_{jc} \\ \hline x_{js} \end{Bmatrix} \tag{42}$$

Referring to equation (31) and making the substitutions in equations (37)-(40), the partitions can be defined.

$$[A^c] = \frac{1}{2}\,[\bar{I}]\Big[[Q^{+-}(a)] + [Q^{--}(a)] + [Q^{++}(a)] + [Q^{-+}(a)] \tag{43}$$

$$[A^s] = \frac{1}{2}\Big[[Q^{+-}(a)] - [Q^{--}(a)] - [Q^{++}(a)] + [Q^{-+}(a)]\Big] \tag{44}$$

$$[B^s] = \frac{1}{2}\,[\bar{I}]\Big[-[Q^{+-}(b)] - [Q^{--}(b)] + [Q^{++}(b)] + [Q^{-+}(b)]\Big] \tag{45}$$

$$[B^c] = \frac{1}{2}\Big[[Q^{+-}(b)] - [Q^{--}(b)] + [Q^{++}(b)] - [Q^{-+}(b)]\Big] \tag{46}$$

Note that since the first rows of $[B^c]$ and $[A^s]$, and the first columns of $[B^s]$ and $[A^s]$ are all zeros, $[P_2(f)]$ is singular. This difficulty can be circumvented in at least two different ways: an a_0 may be inserted in the upper-left corner of $[A^s]$, or the offending rows and columns may be removed.

Once developed, this approach to multiplying to Fourier series is easy to use and highly adaptable to computer applications. When dealing with relatively short series, a method such as this one is probably unnecessary. However, when a product is formed from two series each of which has many terms, or from more than two series, this methodology is extremely useful.

Products of multiple Fourier series- In the preceding discussion, the steps required to form the Fourier product matrix were described. These steps are: (1) expand the product in summation notation, equation (31); (2) define the Q-matrices, equations (33)-(35); (3) combine the Q-matrices to define the cosine and sine substitutions, equations (37)-(40); and (4) partition the product matrix and define the partitions using steps 1 and 3. In forming the products of multiple Fourier series, the same steps can be followed to form arrays that can be used to multiply the series together using simple tensor-like operations.

The first step of expanding the product using summation notation is the only one that requires any extensive hand calculations. This expansion is necessary in order to determine the signs of the cosines and sines for the substitutions that are made in step 4. As an example, consider the three Fourier series, $f(\psi)$, $X(\psi)$, $Y(\psi)$, and their product $Z(\psi)$, where the coefficients of f are known and I, J, K, and N are nonnegative integers.

$$f(\psi) = \sum_{i=0}^{I} [a_i \cos(i\psi) + b_i \sin(i\psi)] \tag{47}$$

$$X(\psi) = \sum_{j=0}^{J} [x_{jc} \cos(j\psi) + x_{js} \sin(j\psi)] \tag{48}$$

$$Y(\psi) = \sum_{k=0}^{K} [y_{kc} \cos(k\psi) + y_{ks} \sin(k\psi)] \tag{49}$$

$$Z(\psi) = \sum_{n=0}^{N} [z_{nc} \cos(n\psi) + z_{ns} \sin(n\psi)] \tag{50}$$

Expanding Z in terms of f, X, and Y,

$$\begin{aligned}
Z(\psi) = \frac{1}{4} \sum_{i=0}^{I} \sum_{j=0}^{J} \sum_{k=0}^{K} \{& a_i x_{jc} y_{kc}[\cos(i + j + k)\psi + \cos(i + j - k)\psi + \cos(i - j + k)\psi] \\
&+ \cos(i - j - k)\psi] + a_i x_{jc} y_{ks}[\sin(i + j + k)\psi - \sin(i + j - k)\psi \\
&+ \sin(i - j + k)\psi - \sin(i - j - k)\psi] + a_i x_{js} y_{kc}[\sin(i + j + k)\psi \\
&+ \sin(i + j - k)\psi - \sin(i - j + k)\psi - \sin(i - j - k)\psi] \\
&+ a_i x_{js} y_{ks}[-\cos(i + j + k)\psi + \cos(i + j - k)\psi + \cos(i - j + k)\psi \\
&- \cos(i - j - k)\psi] + b_i x_{jc} y_{kc}[\sin(i + j + k)\psi + \sin(i + j - k)\psi \\
&+ \sin(i - j + k)\psi + \sin(i - j - k)\psi] + b_i x_{jc} y_{ks}[-\cos(i + j + k)\psi \\
&+ \cos(i + j - k)\psi - \cos(i - j + k)\psi + \cos(i - j - k)\psi] \\
&+ b_i x_{js} y_{kc}[-\cos(i + j + k)\psi - \cos(i + j - k)\psi + \cos(i - j + k)\psi \\
&+ \cos(i - j - k)\psi] + b_i x_{js} y_{ks}[-\sin(i + j + k)\psi + \sin(i + j - k)\psi \\
&+ \sin(i - j + k)\psi - \sin(i - j - k)\psi]\}
\end{aligned} \tag{51}$$

In step 2, where the Q-arrays are defined, the process of identifying like harmonics is begun. To form the Fourier product of M series, 2^M Q-arrays are required. The first step in defining these arrays is to write all of the possible combinations of indices $i, j, k, \ldots$ and set them equal to n. Then, solve each expression for i, which is the subscript of each element of the Q-arrays. For the example of the product of three series, the possible index combinations are: $n = i + j + k$, $n = -(i + j + k)$, $n = i + j - k$, $n = -(i + j - k)$, $n = i - j + k$, $n = -(i - j + k)$, $n = i - j - k$, and $n = -(i - j - k)$. The eight Q-arrays (all three-dimensional) are then

$$Q_{njk}^{+--}(\alpha) = \begin{cases} \alpha_{n-j-k} & ; \quad 0 \le n - j - k \le I \\ 0 & ; \quad n - j - k < 0,\ n - j - k > I \end{cases} \Bigg\} i + j + k = n \tag{52}$$

$$Q_{njk}^{---}(\alpha) = \begin{cases} \alpha_{-n-j-k} & ; \quad 0 \le -n - j - k \le I \\ 0 & ; \quad -n - j - k < 0 \end{cases} \Bigg\} -(i + j + k) = n \tag{53}$$

$$Q_{njk}^{+-+}(\alpha) = \begin{cases} \alpha_{n-j+k} & ; \quad 0 \le n - j + k \le I \\ 0 & ; \quad n - j + k < 0,\ n - j + k > I \end{cases} \Bigg\} i + j - k = n \tag{54}$$

$$Q_{njk}^{--+}(\alpha) = \begin{cases} \alpha_{-n-j+k} & ; \quad 0 \le -n - j + k \le I \\ 0 & ; \quad -n - j + k < 0,\ -n - j + k > I \end{cases} \Bigg\} -(i + j - k) = n \tag{55}$$

$$Q_{njk}^{++-}(\alpha) = \begin{cases} \alpha_{n+j-k} & ; \quad 0 \le n + j - k \le I \\ 0 & ; \quad n + j - k < 0,\ n + j - k > I \end{cases} \Bigg\} i - j + k = n \tag{56}$$

$$Q_{njk}^{-+-}(\alpha) = \begin{cases} \alpha_{-n+j-k} & ; \quad 0 \le -n + j - k \le I \\ 0 & ; \quad -n + j - k < 0,\ -n + j - k > I \end{cases} \Bigg\} -(i - j + k) = n \tag{57}$$

$$Q_{njk}^{+++}(\alpha) = \begin{cases} \alpha_{n+j+k} & ; \quad 0 \le n + j + k \le I \\ 0 & ; \quad n + j + k > I \end{cases} \Bigg\} i - j - k = n \tag{58}$$

$$Q_{njk}^{-++}(\alpha) = \begin{cases} \alpha_{-n+j+k} & ; \quad 0 \le n + j + k \le I \\ 0 & ; \quad -n + j + k < 0,\ -n + j + k > I \end{cases} \Bigg\} -(i - j - k) = n \tag{59}$$

Note that these eight arrays were defined by inspection. There was no need to write out each one to check its contents, because they all follow a specific pattern which defines their use.

Step three completes the identification of like harmonics by combining the Q-matrices to obtain the cosines and sines of each harmonic. Again, this can be done by inspection. Starting with $\cos[(i + j + k + \ldots)\psi]$, the Q-arrays that represent $n = (i + j + k + \ldots)$ are added together and premultiplied by a correction matrix which multiplies all of the $n = 0$ terms by one-half. Similarly, the other cosine functions are obtained by adding the appropriate Q-arrays and premultiplying by the correction matrix. The sine functions are defined by subtracting the Q-arrays from one another and no correction is required. Returning to the example of the product of three series, the cosine and sine expressions are

$$\sum_{i=0}^{I} \alpha_i \cos(i + j + k)\psi = \sum_{m=0}^{N} \bar{I}_{nm}[Q_{mjk}^{+--}(\alpha) + Q_{mjk}^{---}(\alpha)] \ ; \quad n = i + j + k \tag{60}$$

$$\sum_{i=0}^{I} \alpha_i \cos(i + j - k)\psi = \sum_{m=0}^{N} \bar{I}_{nm}[Q^{+-+}_{mjk}(\alpha) + Q^{--+}_{mjk}(\alpha)] \ ; \qquad n = i + j - k \tag{61}$$

$$\sum_{i=0}^{I} \alpha_i \cos(i - j + k)\psi = \sum_{m=0}^{N} \bar{I}_{nm}[Q^{++-}_{mjk}(\alpha) + Q^{-+-}_{mjk}(\alpha)] \ ; \qquad n = i - j + k \tag{62}$$

$$\sum_{i=0}^{I} \alpha_i \cos(i - j - k)\psi = \sum_{m=0}^{N} \bar{I}_{nm}[Q^{+++}_{mjk}(\alpha) + Q^{-++}_{mjk}(\alpha)] \ ; \qquad n = i - j - k \tag{63}$$

$$\sum_{i=0}^{I} \alpha_i \sin(i + j + k)\psi = Q^{+--}_{njk}(\alpha) - Q^{---}_{njk}(\alpha) \ ; \qquad n = i + j + k \tag{64}$$

$$\sum_{i=0}^{I} \alpha_i \sin(i + j - k)\psi = Q^{+-+}_{njk}(\alpha) - Q^{--+}_{njk}(\alpha) \ ; \qquad n = i + j - k \tag{65}$$

$$\sum_{i=0}^{I} \alpha_i \sin(i - j + k)\psi = Q^{++-}_{njk}(\alpha) - Q^{-+-}_{njk}(\alpha) \ ; \qquad n = i - j + k \tag{66}$$

$$\sum_{i=0}^{I} \alpha_i \sin(i - j - k)\psi = Q^{+++}_{njk}(\alpha) - Q^{-++}_{njk}(\alpha) \ ; \qquad n = i - j - k \tag{67}$$

where α may represent either a or b, and where $\bar{I}$ is the correction matrix defined in equation (41)

The final step in forming the product of multiple Fourier series is to partition the product array $[P_M(f)]$ and define the partitions. Within $[P_M(f)]$ the cosine and sine coefficients are grouped together in blocks which are arranged in a checkerboard fashion throughout the array. If the partitions are chosen such that each one contains only cosine or sine coefficients of $f(\psi)$, there will be 2^M partitions. Writing the relationship among $Z(\psi)$, $f(\psi)$, $X(\psi)$, and $Y(\psi)$ for the example of a product of three series in pseudo-matrix notation,

$$\begin{Bmatrix} z_{nc} \\ z_{ns} \end{Bmatrix} = \left[[P_3(f)] \begin{Bmatrix} x_{jc} \\ x_{js} \end{Bmatrix} \right] \begin{Bmatrix} y_{kc} \\ y_{ks} \end{Bmatrix} \tag{68}$$

Partitioning $[P_3(f)]$,

$$z_{nc} = \sum_{k=0}^{K} \left[\sum_{j=0}^{J} (A^{cc}_{njk} x_{jc} + B^{sc}_{njk} x_{js}) y_{kc} + \sum_{j=0}^{J} (B^{cs}_{njk} x_{jc} + A^{ss}_{njk} x_{js}) y_{ks} \right] \tag{69}$$

$$z_{ns} = \sum_{k=0}^{K} \left[\sum_{j=0}^{J} (B^{cc}_{njk} x_{jc} + A^{sc}_{njk} x_{js}) y_{kc} + \sum_{j=0}^{J} (A^{cs}_{njk} x_{jc} + B^{ss}_{njk} x_{js}) y_{ks} \right] \tag{70}$$

Using equation (51) to define the partitions and substituting equations (60)-(67),

$$A_{njk}^{cc} = \frac{1}{4}\sum_{m=0}^{N} \bar{I}_{nm}[Q_{mjk}^{+--}(a) + Q_{mjk}^{---}(a) + Q_{mjk}^{+-+}(a) + Q_{mjk}^{--+}(a) + Q_{mjk}^{++-}(a)$$

$$+ Q_{mjk}^{-+-}(a) + Q_{mjk}^{+++}(a) + Q_{mjk}^{-++}(a)] \tag{71}$$

$$A_{njk}^{cs} = \frac{1}{4}[Q_{njk}^{+--}(a) - Q_{njk}^{---}(a) - Q_{njk}^{+-+}(a) + Q_{njk}^{--+}(a) + Q_{njk}^{++-}(a) - Q_{njk}^{-+-}(a)$$

$$- Q_{njk}^{+++}(a) + Q_{njk}^{-++}(a)] \tag{72}$$

$$A_{njk}^{sc} = \frac{1}{4}[Q_{njk}^{+--}(a) - Q_{njk}^{---}(a) + Q_{njk}^{+-+}(a) - Q_{njk}^{--+}(a) - Q_{njk}^{++-}(a) + Q_{njk}^{-+-}(a)$$

$$- Q_{njk}^{+++}(a) + Q_{njk}^{-++}(a)] \tag{73}$$

$$A_{njk}^{ss} = \frac{1}{4}\sum_{m=0}^{N} \bar{I}_{nm}[-Q_{mjk}^{+--}(a) - Q_{mjk}^{---}(a) + Q_{mjk}^{+-+}(a) + Q_{mjk}^{--+}(a) + Q_{mjk}^{++-}(a)$$

$$+ Q_{mjk}^{-+-}(a) - Q_{mjk}^{+++}(a) - Q_{mjk}^{-++}(a)] \tag{74}$$

$$B_{njk}^{cc} = \frac{1}{4}[Q_{njk}^{+--}(b) - Q_{njk}^{---}(b) + Q_{njk}^{+-+}(b) - Q_{njk}^{--+}(b) + Q_{njk}^{++-}(b) - Q_{njk}^{-+-}(b)$$

$$+ Q_{njk}^{+++}(b) - Q_{njk}^{-++}(b)] \tag{75}$$

$$B_{njk}^{cs} = \frac{1}{4}\sum_{m=0}^{N} \bar{I}_{nm}[-Q_{mjk}^{+--}(b) - Q_{mjk}^{---}(b) + Q_{mjk}^{+-+}(b) + Q_{mjk}^{--+}(b) - Q_{mjk}^{++-}(b)$$

$$- Q_{mjk}^{-+-}(b) + Q_{mjk}^{+++}(b) + Q_{mjk}^{-++}(b)] \tag{76}$$

$$B_{njk}^{sc} = \frac{1}{4}\sum_{m=0}^{N} \bar{I}_{nm}[-Q_{mjk}^{+--}(b) - Q_{mjk}^{---}(b) - Q_{mjk}^{+-+}(b) - Q_{mjk}^{--+}(b) + Q_{mjk}^{++-}(b)$$

$$+ Q_{mjk}^{-+-}(b) + Q_{mjk}^{+++}(b) + Q_{mjk}^{-++}(b)] \tag{77}$$

$$B_{njk}^{ss} = \frac{1}{4}[-Q_{njk}^{+--}(b) + Q_{njk}^{---}(b) + Q_{njk}^{+-+}(b) - Q_{njk}^{--+}(b) + Q_{njk}^{++-}(b) - Q_{njk}^{-+-}(b)$$

$$- Q_{njk}^{+++}(b) + Q_{njk}^{-++}(b)] \tag{78}$$

Having obtained the partitions of the product array, they can be substituted into equations (69) and (70), and $[P_3(f)]$ can be formed. Again, care must be taken in using $[P_3(f)]$ because each "slice" of that array ($j = 1, 2, \ldots J$ or $k = 1, 2, \ldots K$) is a singular matrix, as described in the section on the Fourier product matrix. As in that case, the problem can be eliminated quite easily.

Forming the product array for products of M Fourier series can be quite simple. However, it is obvious that the more series that are multiplied together, the more cumbersome this process becomes. Fortunately, once the product array has been derived for M series, it need never be done again.

Multiblade Coordinate Transformation

The purpose of the multiblade transformation is to transform equations from a coordinate system which rotates at a constant angular velocity to a nonrotating coordinate system. In reference 5 the mathematical definition of multiblade coordinates is presented, but not thoroughly explained. Reference 6 contains a more complete development of the mathematics of the transformation and its use. Since the multiblade transformation is really only a modified Fourier series, the operations that were defined in the preceding sections should be applicable after making some minor modifications. It will be seen that using certain Fourier operations makes the transformation simple to perform for any number of blades, with no <u>ad hoc</u> additions or corrections.

Consider the linear (or linearized), second-order, differential equations for a dynamic system having b identical, equally spaced, time-lagging components. The kth component has a degree of freedom q_k which is referenced to the rotating coordinate system. There is also a degree of freedom p which is referenced to the nonrotating system and is common to all b components. In the rotating system, the equations of motion for q_k and p are respectively

$$M_{qq}(\psi_k)\ddot{q}_k + C_{qq}(\psi_k)\dot{q}_k + K_{qq}(\psi_k)\dot{q}_k + M_{qp}(\psi_k)\ddot{p} + C_{qp}(\psi_k)\dot{p} + K_{qp}(\psi_k)p = F_q(\psi_k) \tag{79}$$

$$M_{pq}(\psi_k)\ddot{q}_k + C_{pq}(\psi_k)\dot{q}_k + K_{pq}(\psi_k)\dot{q}_k + M_{pp}(\psi_k)\ddot{p} + C_{pp}(\psi_k)\dot{p} + K_{pp}(\psi_k)p = F_p(\psi_k) \tag{80}$$

where the coefficients of the equations are periodic in $\psi_k = \psi_0 + 2\pi(k - 1)/b$.

If these equations were to be put into the form of a variational statement, equation (79) would be multiplied by δq_k and equation (80) would be multiplied by δp. The resulting variational statement is

$$\int_{t_1}^{t_2} \{[M_{qq}\ddot{q}_k + C_{qq}\dot{q}_k + K_{qq}q_k + M_{qp}\ddot{p} + C_{qp}\dot{p} + K_{qp}p - F_q]\delta q_k$$

$$+ M_{pq}\ddot{q}_k + C_{pq}\dot{q}_k + K_{pq}q_k + M_{pp}\ddot{p} + C_{pp}\dot{p} + K_{pp}p - F_p]\delta p\}dt = 0 \tag{81}$$

To transform equation (81) from the rotating system into the nonrotating system, only δq_k, q_k, and its derivatives need to be transformed. The degree-of-freedom p is already referenced to the nonrotating coordinate system. Since derivatives of the coordinate q_k would be difficult to manipulate, the dummy variables α_k and β_k

will be substituted for q_k and q_k, respectively. Applying the definition of the multiblade coordinates (ref. 5),

$$\alpha_k = \alpha_0 + \alpha_d(-1)^k + \sum_{n=1}^{N} [\alpha_{nc} \cos(n\psi_k) + \alpha_{ns} \sin(n\psi_k)] \tag{82}$$

$$\beta_k = \beta_0 + \beta_d(-1)^k + \sum_{n=1}^{N} [\beta_{nc} \cos(n\psi_k) + \beta_{ns} \sin(n\psi_k)] \tag{83}$$

$$q_k = q_0 + q_d(-1)^k + \sum_{n=1}^{N} [q_{nc} \cos(n\psi_k) + q_{ns} \sin(n\psi_k)] \tag{84}$$

$$\delta q_k = \delta q_0 + \delta q_d(-1)^k + \sum_{n=1}^{N} [\delta q_{nc} \cos(n\psi_k) + \delta q_{ns} \sin(n\psi_k)] \tag{85}$$

Since the coefficients of equation (81) are periodic functions (i.e., Fourier series), substituting equations (82)-(85) into that equation results in terms that are products of modified Fourier series. The terms $M_{qq}\alpha_k\delta q_k$, $C_{qq}\beta_k\delta q_k$, and $K_{qq}q_k\delta q_k$ are products of three series, $M_{pp}p\delta p$, $C_{pp}p\delta p$, $K_{pp}p\delta p$, and $F_p\delta p$ are single series, and all of the others are products of two series.

To form the products required in equation (81), the same basic methodology described in the preceding section can be applied. The difference lies in the treatment of the $(-1)^k$ terms. Since the locations in the arrays allotted to the $\sin(0\psi)$ terms are always filled with zeros, those locations may be used for the $(-1)^k$ terms. This also preserves the symmetry of the arrays.

The formation of the modified Fourier products in equation (81) performs the transformation from the rotating to the nonrotating coordinate system. To complete the process, the dummy variables α_k and β_k need to be eliminated. Using the definition of the derivative of a series having periodic coefficients

$$\begin{Bmatrix} \alpha_0 \\ \alpha_{1c} \\ \alpha_{2c} \\ \cdot \\ \cdot \\ \cdot \\ \alpha_{nc} \\ \alpha_d \\ \alpha_{1s} \\ \alpha_{2s} \\ \cdot \\ \cdot \\ \cdot \\ \alpha_{ns} \end{Bmatrix} = \left[[I] \frac{\partial^2}{\partial \psi_k} + 2[D] \frac{\partial}{\partial \psi_k} + [D^2] \right] \begin{Bmatrix} q_0 \\ q_{1c} \\ q_{2c} \\ \cdot \\ \cdot \\ \cdot \\ q_{nc} \\ q_d \\ q_{1s} \\ q_{2s} \\ \cdot \\ \cdot \\ \cdot \\ q_{ns} \end{Bmatrix} \tag{86}$$

$$\begin{Bmatrix} \beta_0 \\ \beta_{1c} \\ \beta_{2c} \\ \cdot \\ \cdot \\ \cdot \\ \beta_{nc} \\ \beta_d \\ \beta_{1s} \\ \beta_{2s} \\ \cdot \\ \cdot \\ \cdot \\ \beta_{ns} \end{Bmatrix} = \left[[I] \frac{\partial}{\partial \psi_k} + [D] \right] \begin{Bmatrix} q_0 \\ q_1 \\ q_2 \\ \cdot \\ \cdot \\ \cdot \\ q_{nc} \\ q_d \\ q_{1s} \\ q_{2s} \\ \cdot \\ \cdot \\ \cdot \\ q_{ns} \end{Bmatrix} \tag{87}$$

Once equations (86) and (87) are substituted into the transformed equations, the transformation is essentially complete, except for sorting the periodic terms.

Thus far, the number of rotating components, b, has not been needed for the transformation. However, it is of great importance in determining the number of multiblade coordinates that are needed, as well as determining the harmonic contributions in the nonrotating system. The number of multiblade coordinates that are required is equal to the number of components, where $N = (b - 2)/2$ for b even and $N = (b - 1)/2$ for b odd. If b is odd, the differential collective mode does not exist (ref. 5). To determine the harmonic contributions in the nonrotating coordinate system, equation (81) is summed over the b components and the identities in equations (5)-(8) are applied. The transformation is now complete, and the variational statement may be broken down into differential equations.

In applying this method for using the multiblade transform, there are shortcuts that can be taken. There is no practical reason to convert the differential equations to a variational statement and then convert back. This was done to show the logic behind the operations that were performed. Similarly, there is no need for the dummy variables α_k and β_k which were included only for convenience. The sorting of harmonics can be done by inspection. If b is odd, only those harmonics that are integer multiples of b are retained. If b is even, harmonics that are integer multiples of b are retained for all terms not involving $(-1)^k$. In those terms where $(-1)^k$ appears, only those harmonics which are odd integer multiples of $b/2$ are retained.

CONCLUDING REMARKS

In this Memorandum, the development of Fourier operations which was begun in reference 1 is extended. Summation relations for expressions involving $(-1)^k$ are defined, the definition of the derivatives of Fourier series are generalized to include series having coefficients that are functions of ψ, and a method for performing the symbolic multiplication of any number of Fourier series is developed. In addition, the operations are applied to the multiblade coordinate transformation.

While the harmonic balance method may not be suitable for solving all types of response problems, its elegance and understandability are desirable features. Its major disadvantages are the large number of algebraic equations that must be solved and the necessity for forming products and derivatives of Fourier series. Because of the work performed in reference 1 and the extensions included in this report, the second disadvantage becomes unimportant since those tasks can be performed by the computer.

Other applications of Fourier operations are possible. In this report, the example of using these operations to perform a coordinate transformation was developed. The result of this development was a procedure which has general applicability, is understandable, and is suitable for automation. Many other applications for these operations may be found.

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